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Applications of Random Walk and Gambler's Ruin on Irreducible Periodic Markov Chain

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Abstract
The Physical or Mathematical behaviour of any system may be represented by describing all the different states it may occupy and by indicating how it moves among these states. In this study, the states of the Markov chain with the integers $0, \pm 1, \pm 2, \ldots$ (the drunkard's straight line) where the only transitions from any state k are to neighbouring states $(k+1)$: a step to the right with probability p and $(k-1)$: a step to the left with probability $q = (1-p)$ has been investigated, in order to provide some insight in determining whether the gambler is ruined, that is, loses all his money in which Markov chain moves to state 0, and taken to be an absorbing state or wins a fortune that Markov chain moves into absorbing state $N > k$, where N is large). Our quest is to analyse the transition diagram and probability transition matrix to obtain the solution to the system of linear equations for the gambler's ruin problem. The theorems, Gaussian elimination method with the help of some existing equations and laws in Markov chain are used. Illustrative example is considered on playing cards and the following probabilities are obtained: The probability that Grace ends up with all the cards, the probability that Gloria ends up with all the cards and the probability that Gloria takes all of Grace's cards.

Keywords: Denumerable Markov chain, gambler's ruin, irreducible Markov chain, random walk, recurrence state

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Introduction
The image of a drunken man trying to walk home from the club along an imagined straight route might be utilized to analyze a normal random walk. That is, he walks zigzag while seeming to walk straight, sometimes to the right and sometimes to the left of his imagined straight line. This colorful scenario leads to the random walk, a common type of Markov chain problem. The integers $0, \pm 1, \pm 2, \ldots$ (the drunkard's straight line) are the states of the Markov chain, and the only transitions from any state k are to neighboring states $(k+1)$ (a step to the right) with probability p and $(k-1)$ (a step to the left) with probability $q = (1-p)$. Figure 1 depicts the state transition diagram, using state 0 as the starting point. The technique is called a symmetric random walk when $\mathrm{p} = 1/2$. The gambler's ruin problem refers to a situation in which the state space is finite. A gambler starts with a particular amount of money, say k Nairas (the process starts in state $\mathrm{k} > 0$ in terms of the Markov chain), and on each play, the gambler can either win or lose a naira with probability ρ or $(1-\rho)$. The goal is to see if the gambler is ruined, that is, if he loses all of his money (the Markov chain travels to state 0, which is considered an absorbing state) or if he wins a

fortune (the Markov chain advances to absorbing state $N > k$, where N is large). The state space is taken to be the set of nonnegative integers, and the Markov chain begins in state 0. Other variations are based on a random walk that is infinite on one side only, the state space is taken to be the set of nonnegative integers, and the Markov chain begins in state 0. The random walk problem can be defined in multiple dimensions. Random walks in two dimensions occur on a grid that can be unlimited and span the entire plain, infinite but only span the upper right quadrant, or finite. Each state communicates with at least four of its neighbors, North, South, East, and West, in these circumstances. When each transition probability is 1/4, the process is symmetric. There is a path from any state to any other state in random walks problems where the Markov chain is irreducible. All of the states are either positive recurrent, null recurrent, or transient in this situation. Except for the symmetric case, $\rho = 1/2$, all states in a one-dimensional random walk on the integers 0,1,2,3, are transient. In two dimensions, the same result applies, but in three dimensions or higher, even in the symmetric situation, all states are transient. Romanovsky (1970) introduced the study, application and simulation of a discrete Markov Chains and this was extended to the introduction of Numerical Solutions of Markov Chains and random walk by Stewart (1994, 2009) while the suitability of the Markov chain approach was demonstrated in the wind feed in Germany by Pesch (2015), and Uzun (2017) Probabilities and reachability matrix in close and open classification group of states in Markov chain are transient, recurrence and communicating. We Consider the infinite carried out the study to predict the direction of the gold price movement and to determine the probabilistic transition matrix of the closing returns of gold prices using Markov chain model of fuzzy state, the application of Markov chain using a data mining approach to get a prediction of the monthly rainfall data is shown by Aziza (2019), the application of Markov chain on the spread of disease infection which shown that Hepatitis B was more infectious overtime than tuberculosis and HIV was demonstrated by Clement (2019) while the application of Markov chain to Journalism was discussed by Vermeer (2020). However, this study, applications of random walk and gambler's ruin on irreducible periodic Markov chain is considered with illustrative examples to obtained its performance measures,

Notations

k, is the number of state; p, the probability of transition from state k to state $(k+1)$; $q = 1-p$, the probability of transition from state k to state $(k-1)$ and x_j,y_j, z_j, $j = 1,2,\cdots,N$, solutions of the system of linear equations.

Materials and Methods

The study area consisted of analysis of random walks and gambler ruin on irreducible periodic Markov chain. We started with the analysis of the state transition diagram for a random walk on the integers, as well as the analysis of its corresponding probability transition diagram. The state transition diagram could be represented as follows in Figure 1.

(denumerable) Markov chain to analyse the probability transition matrix in random walk as given in Figure 2 below:

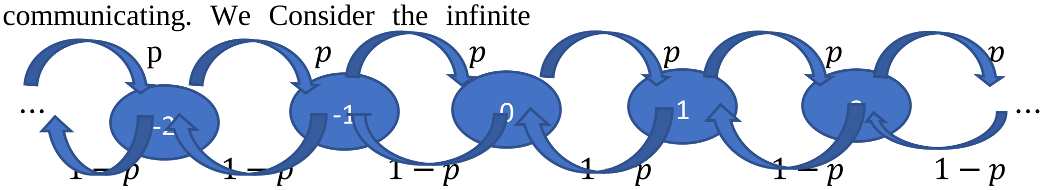

Figure 1: State transition diagram for a random walk on the integers

$$P = \begin{pmatrix} 0 & 1 & 1 & \cdots & & \\ q & 0 & p & 0 & \cdots & \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Figure 2: State transition probability matrix for a random walk on the integers

where p is a positive probability and $q = 1 - p$. Observe that every time the Markov chain reaches state 0, it must leave it again at the next time step. State 0 is said to constitute a reflecting barrier while It may also be more appropriate to set $p_{11} = 1 - p$ and $p_{12} = p$, called a Bernoulli barrier, or $p_{11} = 1$ and $p_{12} = 0$, called an absorbing barrier and we pursue our analysis with the reflecting barrier option. Since every state can reach every other state, the Markov chain is irreducible and hence all the states are positive recurrent or all the states are null recurrent or all the states are transient. We also notice that a return to any state is possible only in a number of steps that is a multiple of 2. Thus, the Markov chain is periodic with period equal to 2. We now state and prove the theorems below to classify the states of this chain, also as a tool for study random walks and gambler's ruin.

Theorem 1: Let P be the single step transition probability matrix of an irreducible Markov chain. Then all the states of this Markov chain are positive recurrent if and only if the system of linear equations

$$y = yP, \qquad (1)$$

in which y is a row vector, has a solution with $\sum_{all\ j} y_j = 1$.

Theorem 2: Let P be the single step transition probability matrix of an irreducible Markov chain. Let P^I be the matrix obtained from P by deleting the row k and column k. Then all states are recurrent if and only if the solution of

$$x = P^I x, \ \ 0 \le x \le 1, \qquad (2)$$

is the zero vector.

To prove the first theorem, consider the system of equation $y = yP$,

$$(y_0 \ \ y_1 \, y_2 \cdots) = (y_0 \ \ y_1 \ \ y_2 \cdots) \begin{pmatrix} 0 & 1 & 1 & \cdots & & \\ q & 0 & p & 0 & \cdots & \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \qquad (3)$$

Taking this equation one at a time, we find

$$y_0 = y_1 q \Rightarrow y_1 = \frac{1}{q} y_0,$$

$$y_1 = y_0 + q y_2 \Rightarrow y_2 = \frac{1}{q}(y_1 - y_0) = \frac{1}{q}(1 - q) y_1 = \left(\frac{p}{q}\right) y_1 = \frac{1}{q}\left(\frac{p}{q}\right) y_0.$$

All subsequent equations are of the form

$$y_j = p y_{-1} + q y_{j+1} \ \text{ for } \ j \ge 2 \qquad (4)$$

and have the solution

$$y_{j+1} = \frac{p}{q} y_j, \qquad (5)$$

which may be proven by induction as follows. Base clause, $j = 2$:

$$y_2 = p y_1 + q y_3 \Rightarrow y_3 = \frac{1}{q}(y_2 - p y_1) = \frac{1}{q} y_2 - y_2 = \left(\frac{p}{q}\right) y_2. \qquad (6)$$

We now assume this solution to hold for *j* and prove it true for $(j + 1)$. From

$$y_j = p y_{j-1} + q y_{j+1}$$

$$y_j = p y_{j-1} + q y_{j+1} \ \text{ for } \ j \ge 2$$

$$y_{j+1} = \frac{1}{q}\left(y_j - p y_{j-1}\right) = \left(\frac{1}{q}\right) y_j - \left(\frac{p}{q}\right) y_{j-1} = \left(1 - \frac{1}{q}\right) y_j = \left(\frac{p}{q}\right) y_j \qquad (7)$$

which completes the proof. This solution allows us to write any component in terms of y_0. We have

$$y_j = \left(\frac{p}{q}\right) y_{j-1} = \left(\frac{p}{q}\right)^2 y_{j-2} = \cdots = \left(\frac{p}{q}\right)^{j-1} y_1 = \frac{1}{q}\left(\frac{p}{q}\right)^{j-1} y_o$$

and summing over all y_j, we find

$$\sum_{j=0}^{\infty} y_j = y_0 \left[1 + \frac{1}{q} \sum_{j=1}^{\infty} \left(\frac{p}{q}\right)^{j-1}\right]. \qquad (8)$$

Since the summation inside the square bracket is finite if and only if $p < q$ in which case

$$\sum_{j=1}^{\infty} \left(\frac{p}{q}\right)^{j-1} = \sum_{j=0}^{\infty} \left(\frac{p}{q}\right)^{j} = \left(\frac{1}{1-p/q}\right) \quad iff \quad p < q. \qquad (9)$$

The sum of all y_j being equal to 1 implies that

$$1 = y_0 \left[1 + \frac{1}{q}\left(\frac{1}{1-p/q}\right)\right]$$

Simplifying the term inside the square brackets, we obtain

$$1 = y_0 \left[1 + \left(\frac{1}{q-p}\right)\right] = y_0 \left(\frac{q-p+1}{q-p}\right) = y_0 \left(\frac{2q}{q-p}\right) \qquad (10)$$

and hence

$$y_0 = \left(\frac{q-p}{2q}\right) = \frac{1}{2}\left(1 - \frac{p}{q}\right) \quad \text{for } p < q. \qquad (11)$$

The remaining y_j are obtained as

$$y_j = \frac{1}{q}\left(\frac{p}{q}\right)^{j-1} y_0 = \frac{1}{2q}\left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^{j-1}. \qquad (12)$$

This solution satisfies the conditions of Theorem 1 (it exists and its components sum to 1) and hence all states are positive recurrent (when $p < q$). We also see that the second part of this theorem is true (all y_j are strictly positive). Finally, this theorem tells us that there is no other solution. We now examine the other possibilities, namely, $p = q$ and $p > q$. Under these conditions, either all states are null recurrent or else all states are transient from Theorem (1). We shall now use Theorem 2 to determine which case holds. Let the matrix P^I be the matrix obtained from P when the first row and column of P is removed and let us consider the system of equations $x = P^I x$. We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & p & 0 & 0 & \cdots \\ q & 0 & p & 0 & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}. \qquad (13)$$

Again taking these equations one at a time, we have

$$x_1 = p x_2,$$

$$x_2 = q x_1 + p x_3,$$

and in general

$$x_j = q x_{j-1} + p x_{j+1}, \qquad (14)$$

Writing the left-hand side as $p x_j + q x_j$, we have

$$p x_j + q x_j = q x_{j-1} + p x_{j+1} \quad \text{for } j \geq 2, \qquad (15)$$

which yields

$$p\left(x_{j+1}-x_j\right)=q\left(x_j-x_{j-1}\right) \quad \text{for } \ j \geq 2. \qquad (16)$$

When j = 1, a similar rearrangement gives

$$p(x_2-x_1)=qx_1.$$

It follows from Equation (16) that

$$x_{j+1}-x_j=\frac{q}{p}\left(x_j-x_{j-1}\right)=\cdots=\left(\frac{q}{p}\right)^{j-1}(x_2-x_1)$$

$$=\left(\frac{q}{p}\right)^{j-1}\left(\frac{q}{p}\right)x_1=\left(\frac{q}{p}\right)^{j}x_1 \ \text{ for } \ j \geq 1. \qquad (17)$$

And

$$x_{j+1}=\left(x_{j+1}-x_j\right)+\left(x_j-x_{j-1}\right)+\left(x_{j-1}-x_{j-2}\right)+\cdots+(x_2-x_1)+x_1$$

$$=\left(\frac{q}{p}\right)^{j}x_1+\left(\frac{q}{p}\right)^{j-1}x_1+\cdots+\left(\frac{q}{p}\right)x_1+x_1\ ,$$

$$=\left[1+\left(\frac{q}{p}\right)+\left(\frac{q}{p}\right)^2+\cdots+\left(\frac{q}{p}\right)^j\right]x_1. \qquad (18)$$

Consider now the case in which p = q. We obtain

$$x_j=jx_1 \qquad (19)$$

In order to satisfy the conditions of Theorem 2, we need to have $0 \leq x_i \leq 1$ for all j. Therefore the only possibility in this case ($p=q$) is that $x_1=0$, in which case $x_j=0$ for all j. Since the only solution is $x=0$, we may conclude from Theorem 2 that all the states are recurrent states and since we know that they are not positive recurrent, they must be null recurrent. Finally, consider the case in which $p>q$. In this case, $\left(\frac{q}{p}\right)$ is a fraction and the summation converge. We have

$$x_j=\left[1+\left(\frac{q}{p}\right)+\left(\frac{q}{p}\right)^2+\cdots+\left(\frac{q}{p}\right)^j\right]x_1=\sum_{k=0}^{j-1}\left(\frac{q}{p}\right)^k x_1=\frac{1-(q/p)^j}{1-q/p}x_1. \qquad (20)$$

It is apparent that if we now set

$$x_1=1-\left(\frac{q}{p}\right)$$

we obtain a value of x_i that satisfies $0 \leq x_i \leq 1$ and, furthermore, we also obtain

$$x_j=1-\left(\frac{q}{p}\right)^j, \qquad (21)$$

which also satisfies $0<x_j<1$ for all $j \geq 0$. From Theorem 2 we may now conclude that all the states are transient when $p>q$.

Results

Let us now consider a different random walk, the gambler's ruin problem. The gambler begins with i Nairas and on each play either wins a Naira with probability p or loses a Naira with probability $q=1-p$. If X_n is the amount of money he has after playing n times, then $\{X_n, n=0,1,2,\cdots\}$ is a Markov chain and the transition probability matrix is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & & \cdots & & & 0 \\ q & 0 & p & 0 & & \cdots & & & 0 \\ 0 & q & 0 & p & & \cdots & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ 0 & & \cdots & & q & 0 & p & & 0 \\ 0 & & \cdots & & 0 & q & 0 & & p \\ 0 & & \cdots & & 0 & 0 & 0 & & 1 \end{pmatrix}$$

Fig 3: Transition probability matrix for gambler's ruin problem

We would like to find the probability that the gambler will eventually make his fortune by arriving in state *N* given that he starts in state i. Let y_i be this probability. We immediately have that $y_0 = 0$ and $y_N = 1$, since the gambler cannot start with zero Naira and win *N* nairas, while if he starts with *N* nairas he has already made his fortune. Given that the gambler starts with i nairas, after the first play he has $(i+1)$ nairas with probability *p*, and $(i-1)$ nairas with probability q. It follows that the probability of ever reaching state *N* from state i is the same as the probability of reaching *N* beginning in $(i+1)$ with probability *p* plus the probability of reaching state *N* beginning in state $(i-1)$ with probability q. In other words

$$y_1 = py_{i+1} + qy_{i-1}$$

This is the equation that will allow us to solve our problem. It holds for all $1 \leq i \leq (N-1)$. We should note that it does not result from the multiplication of a vector by the transition probability matrix, but rather by conditioning on the result of the first play. Writing this equation as

$$y_1 = py_i + qy_i = y_1 = py_{i+1} + qy_{i-1},$$

allows us to derive the recurrence relation

$$y_{i+1} - y_i = \frac{q}{p}(y_i - y_{i-1}), \quad i = 1,2,\cdots,N-1, \qquad (22)$$

Now, using the fact that $y_0 = 0$, we have

$$y_2 - y_1 = \frac{q}{p}(y_1 - y_0) = \frac{q}{p}y_1 ,$$

$$y_3 - y_2 = \frac{q}{p}(y_2 - y_1) = \left(\frac{q}{p}\right)y_1 ,$$

$$\vdots$$

$$y_i - y_{i-1} = \frac{q}{p}(y_{i-1} - y_{i-2}), \quad i = 1, 2, \cdots, N \qquad (23)$$

Adding these equations, we find

$$y_i - y_1 = \left(\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)y_1,$$

i.e.,

$$y_i = \left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)y_1 = \sum_{k=0}^{i-1}\left(\frac{q}{p}\right)^k y_1. \qquad (24)$$

We must consider the two possible cases for this summation, $p \neq q$ and $p = q = 1/2$. When $p \neq q$, then

$$y_i = \left(\frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}}\right) y_1, \tag{25}$$

And in particular

$$\left(\frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}}\right) y_1 = y_N = 1. \tag{26}$$

This allows us to compute y_1 as

$$y_1 = \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N}, \tag{27}$$

and from this, all the remaining values of $y_i, i = 1, 2, 3, \cdots, N-1$, can be found. Collecting these results together, we have

$$y_i = \left(\frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}}\right) \times \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} = \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}, \quad i = 1, 2, \cdots, N \quad and\ p \neq q. \tag{28}$$

Notice the limits as *N* tends to infinity:

$$\lim_{N\to\infty} y_i = \begin{cases} 1-\left(\frac{q}{p}\right)^i, & p > 1/2, \\ 0, & p < 1/2. \end{cases} \tag{29}$$

Illustrative Example 1: Grace and her sister Gloria play cards. At each game Grace has a 58% chance of winning (perhaps he cheats) while Gloria has only a 42% chance of winning. If Grace starts with 14 cards and Gloria with 18, what is the probability that Grace ends up with all the cards? Suppose Grace starts with only 6 marbles (and Gloria with 15), what is the probability that Gloria ends up with all the cards?

We shall use the following equation generated during the gambler's ruin problem:

$$y_i = \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}, \quad i = 1, 2, \cdots, N \quad and\ p \neq q.$$

Substituting in the values $P = 0.58$, $i = 14$, and $N = 32$,, we have

$$y_{14} = \frac{1-\left(\frac{0.42}{0.58}\right)^{14}}{1-\left(\frac{0.42}{0.58}\right)^{32}} = 0.9891.$$

To answer the second part, the probability that Gerard, in a more advantageous initial setup, wins

all the marbles, we first find y_6, the probability that Grace wins them all. We have

$$y_6 = \frac{1-\left(\frac{0.42}{0.58}\right)^{6}}{1-\left(\frac{0.42}{0.58}\right)^{21}} = 0.9999$$

So, the probability that Grace takes all of Gloria's cards is only 1 − 0.99999 = 0.00001

Discussion

This means that, when the game favours the gambler $(p > 1/2)$, there is a positive probability that the gambler will make his fortune. However, when the game favours the house, then the gambler is sure to lose all his money. This same sad result holds when the game favours neither the house nor the gambler $(p = 1/2)$. When $(p = q = 1/2)$, then $y_i = 1/N$, for $i = 1, 2, \cdots, N$ and hence y_i approaches zero as $N \to \infty$. Also, in the illustrative example, the probability that

Grace ends up with all the cards is given as 0.9891, and the probability that Gloria ends up with all the cards is 0.99999, while the the probability that Gloria takes all of Grace's cards is 0.0001.

Conclusion

The states of the Markov chain with the integers $0, \pm 1, \pm 2, \ldots$ (the drunkard's straight line) where the only transitions from any state k are to neighbouring states $(k+1)$: a step to the right with probability p and $(k-1)$: a step to the left with probability $q = (1-p)$ has been investigated, in order to provide some insight in determining whether the gambler is ruined, that is, loses all his money in which the Markov chain moves to state 0, and taken to be an absorbing state or wins a fortune that Markov chain moves into absorbing state $N > k$, where N is large. Our quest is to analyse the transition diagram and probability transition matrix to obtain the solution to the system of linear equations for the gambler's ruin problem. The theorems, Gaussian elimination method with the help of some existing equations in Markov chain are used. The probability that Grace ends up with all the cards is given as 0.9891, and the probability that Gloria ends up with all the cards is 0.99999, while the the probability that Gloria takes all of Grace's cards is 0.0001.

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